

Geometrical Objects on the First Order Jet Space $J^1(T, \mathbb{R}^5)$ Produced by the Lorenz Atmospheric DEs System

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Abstract

The aim of this paper is to construct natural geometrical objects on the 1-jet space $J^1(T, \mathbb{R}^5)$, where $T \subset \mathbb{R}$, like a non-linear connection, a generalized Cartan connection, together with its d-torsions and d-curvatures, a jet electromagnetic d-field and a jet Yang-Mills energy, starting from the given Lorenz atmospheric DEs system and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ on $T \times \mathbb{R}^5$.

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1 Jet Riemann-Lagrange geometry produced by a first order non-linear DEs system

Many authors, like Asanov [1], Saunders [8] and many others, studied the contravariant differential geometry of 1-jet spaces. Going on with the geometrical studies of Asanov and using as a pattern the Lagrangian geometrical ideas developed by Miron and Anastasiei [4], the author of this paper has developed the *Riemann-Lagrange geometry of 1-jet spaces* [5], which is very suitable for the geometrical study of the solutions of a given DEs or PDEs system, via the *least squares variational method* proposed by Udriște and Neagu in [7], [9].

In what follows we present the main jet Riemann-Lagrange geometrical results that, in author opinion, characterize a given non-linear DEs system of order one. In this direction, let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \rightarrow T \times \mathbb{R}^n, \quad n \geq 2,$$

whose local coordinates (t, x^i, x_1^i) , $i = \overline{1, n}$, transform by the rules

$$\tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{x}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot x_1^j.$$

Remark 1.1 *From a physical point of view, the coordinate t has the physical meaning of **relativistic time**, the coordinates $(x^i)_{i=\overline{1, n}}$ represent **spatial coordinates** and the coordinates $(x_1^i)_{i=\overline{1, n}}$ have the physical meaning of **relativistic velocities**.*

Let us consider that $X = \left(X_{(1)}^{(i)}(x^k) \right)$ is an arbitrary d-tensor field on the 1-jet space $J^1(T, \mathbb{R}^n)$, whose local components transform by the rules

$$\tilde{X}_{(1)}^{(i)} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot X_{(1)}^{(j)}.$$

It is obvious that the d-tensor field X produces the jet first order DEs system (*jet dynamical system*)

$$x_1^i = X_{(1)}^{(i)}(x^k(t)), \quad \forall i = \overline{1, n}, \quad (1.1)$$

where $c(t) = (x^i(t))$ is an unknown curve on \mathbb{R}^n (i. e., a jet field line of the d-tensor field X) and we used the notation

$$x_1^i \stackrel{\text{not}}{=} \dot{x}^i = \frac{dx^i}{dt}, \quad \forall i = \overline{1, n}.$$

Automatically, the jet first order DEs system (1.1), together with the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$ on $T \times \mathbb{R}^n$, produces the *jet least squares Lagrangian function*

$$JLS_{\Delta}^{\text{DEs}} : J^1(T, \mathbb{R}^n) \rightarrow \mathbb{R}_+,$$

expressed by

$$\begin{aligned} JLS_{\Delta}^{\text{DEs}}(x^k, x_1^k) &= \sum_{i,j=1}^n \delta_{ij} \left[x_1^i - X_{(1)}^{(i)}(x) \right] \left[x_1^j - X_{(1)}^{(j)}(x) \right] = \\ &= \sum_{i=1}^n \left[x_1^i - X_{(1)}^{(i)}(x) \right]^2, \end{aligned} \quad (1.2)$$

where $x = (x^k)_{k=\overline{1, n}}$. Because the *global minimum points* of the *jet least squares energy action*

$$\mathbb{E}_{\Delta}^{\text{DEs}}(c(t)) = \int_a^b JLS_{\Delta}^{\text{DEs}}(x^k(t), \dot{x}^k(t)) dt$$

are exactly the solutions of class C^2 of the jet first order DEs system (1.1), it follows that we may regard the jet least squares Lagrangian function $JLS_{\Delta}^{\text{DEs}}$ as a natural geometrical substitut for the DEs system of order one (1.1), on the 1-jet space $J^1(T, \mathbb{R}^n)$.

Remark 1.2 *It is important to note that any solution of class C^2 of the jet first order DEs system (1.1) verify the second order Euler-Lagrange equations produced by the jet least squares Lagrangian function JLS_{Δ}^{DEs} (**jet geometric dynamics**)*

$$\frac{\partial [JLS_{\Delta}^{DEs}]}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial [JLS_{\Delta}^{DEs}]}{\partial \dot{x}^i} \right) = 0, \quad \forall i = \overline{1, n}. \quad (1.3)$$

Conversely, this statement is not true because there exist solutions for the Euler-Lagrange DEs system (1.3) which are not global minimum points for the jet least squares energy action $\mathbb{E}_{\Delta}^{DEs}$, that is which are not solutions for the jet first order DEs system (1.1).

But, a Riemann-Lagrange geometry on $J^1(T, \mathbb{R}^n)$ produced by the jet least squares Lagrangian function JLS_{Δ}^{DEs} , via its Euler-Lagrange equations (1.3), geometry in the sense of non-linear connection, generalized Cartan connection, d-torsions, d-curvatures, jet electromagnetic field and jet Yang-Mills energy, is now completely done in the papers [5], [6] and [7]. For that reason, we introduce the following concept:

Definition 1.3 *Any geometrical object on $J^1(T, \mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function JLS_{Δ}^{DEs} , via its second order Euler-Lagrange equations (1.3), is called **geometrical object produced by the jet first order DEs system (1.1)**.*

In a such context, we give the following geometrical result, which is proved in [6] and, for the multi-time general case, in [7]. For all details, the reader is invited to consult the book [5].

Theorem 1.4 *(i) The canonical non-linear connection on $J^1(T, \mathbb{R}^n)$ produced by the jet first order DEs system (1.1) has the local components*

$$\Gamma^{DEs} = \left(M_{(1)1}^{(i)}, N_{(1)j}^{(i)} \right),$$

where

$$M_{(1)1}^{(i)} = 0 \text{ and } N_{(1)j}^{(i)} = -\frac{1}{2} \left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i} \right], \quad \forall i, j = \overline{1, n}.$$

(ii) All adapted components of the canonical generalized Cartan connection CT^{DEs} produced by the jet first order DEs system (1.1) vanish.

(iii) The effective adapted components $R_{(1)jk}^{(i)}$ of the torsion d-tensor T^{DEs} of the canonical generalized Cartan connection CT^{DEs} produced by the jet first order DEs system (1.1) are

$$R_{(1)jk}^{(i)} = -\frac{1}{2} \left[\frac{\partial^2 X_{(1)}^{(i)}}{\partial x^k \partial x^j} - \frac{\partial^2 X_{(1)}^{(j)}}{\partial x^k \partial x^i} \right], \quad \forall i, j, k = \overline{1, n}.$$

(iv) All adapted components of the **curvature d-tensor** R^{DEs} of the canonical generalized Cartan connection CT^{DEs} **produced by the jet first order DEs system (1.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the jet first order DEs system (1.1)** has the expression

$$F^{DEs} = F_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + N_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1, n},$$

and

$$F_{(i)j}^{(1)} = \frac{1}{2} \left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i} \right], \quad \forall i, j = \overline{1, n}.$$

(vi) The adapted components $F_{(i)j}^{(1)}$ of the electromagnetic d-form F^{DEs} produced by the jet first order DEs system (1.1) verify the **generalized Maxwell equations**

$$\sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0,$$

where $\sum_{\{i,j,k\}}$ represents a cyclic sum and

$$F_{(i)j||k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k}$$

has the geometrical meaning of the horizontal local covariant derivative produced by the Berwald linear connection $B\Gamma_0$ on $J^1(T, \mathbb{R}^n)$. For more details, please consult [5].

(vii) The **geometric jet Yang-Mills energy produced by the jet first order DEs system (1.1)** is defined by the formula

$$EYM^{DEs}(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left[F_{(i)j}^{(1)} \right]^2.$$

Remark 1.5 If we use the following matriceal notations

- $J(X_{(1)}) = \left(\frac{\partial X_{(1)}^{(i)}}{\partial x^j} \right)_{i,j=\overline{1,n}}$ - the **Jacobian matrix**,
- $N_{(1)} = \left(N_{(1)j}^{(i)} \right)_{i,j=\overline{1,n}}$ - the **non-linear connection matrix**,
- $R_{(1)k} = \left(R_{(1)jk}^{(i)} \right)_{i,j=\overline{1,n}}, \quad \forall k = \overline{1, n}$, - the **torsion matrices**,
- $F^{(1)} = \left(F_{(i)j}^{(1)} \right)_{i,j=\overline{1,n}}$ - the **electromagnetic matrix**,

then the following matriceal geometrical relations attached to the jet first order DEs system (1.1) hold good:

1. $N_{(1)} = -\frac{1}{2} [J(X_{(1)}) - {}^T J(X_{(1)})],$
2. $R_{(1)k} = \frac{\partial}{\partial x^k} [N_{(1)}], \forall k = \overline{1, n},$
3. $F^{(1)} = -N_{(1)},$
4. $EYM^{DEs}(x) = \frac{1}{2} \cdot \text{Trace} [F^{(1)} \cdot {}^T F^{(1)}],$ that is the jet Yang-Mills energy coincides with the norm of the skew-symmetric electromagnetic matrix $F^{(1)}$ in the Lie algebra $\mathfrak{o}(n) = L(O(n)).$

In the sequel, we apply the above jet contravariant Riemann-Lagrange geometrical results to the Lorenz five-components atmospheric DEs system introduced by Lorenz [3] and studied, via the Melnikov function method for Hamiltonian systems on Lie groups, by Birtea, Puta, Rațiu and Tudoran [2].

2 Jet Riemann-Lagrange geometry produced by the Lorenz simplified model of Rossby gravity wave interaction

The first model equations for the atmosphere are that so called *primitive equations*. It seems that this model produces wave-like motions on different time scales:

- on the one hand, this model produces the slow motions which have a period of order of days (these slow-waves are called *Rossby waves*);
- on the other hand, this model produces fast motions which have a period of hours (these fast-waves are called *gravity waves*).

The question of how to balance these two time scales leads Lorenz [3] to consider a simplified version of the primitive equations model, which is given by the following non-linear system of five differential equations [2]:

$$\left\{ \begin{array}{l} \frac{dx^1}{dt} = -x^2x^3 + \varepsilon x^2x^5 \\ \frac{dx^2}{dt} = x^1x^3 - \varepsilon x^1x^5 \\ \frac{dx^3}{dt} = -x^1x^2 \\ \frac{dx^4}{dt} = -x^5 \\ \frac{dx^5}{dt} = x^4 + \varepsilon x^1x^2, \end{array} \right. \quad (2.1)$$

where the variables x^4, x^5 represent the fast gravity wave oscillations and the variables x^1, x^2, x^3 are the slow Rossby wave oscillations, with a parameter ε which is related to the physical Rossby number.

Remark 2.1 *It is obvious that, from a physical point of view, the Lorenz atmospheric DEs system (2.1) couples the Rossby waves with the gravity waves.*

Naturally, the Lorenz atmospheric DEs system (2.1) can be regarded as a non-linear DEs system of order one on the 1-jet space $J^1(T, \mathbb{R}^5)$, which is produced by the d-tensor field $X = \left(X_{(1)}^{(i)}(x) \right)$, where $i = \overline{1, 5}$ and

$$x = (x^1, x^2, x^3, x^4, x^5),$$

having the local components

$$\begin{aligned} X_{(1)}^{(1)}(x) &= -x^2x^3 + \varepsilon x^2x^5, \\ X_{(1)}^{(2)}(x) &= x^1x^3 - \varepsilon x^1x^5, \\ X_{(1)}^{(3)}(x) &= -x^1x^2, \\ X_{(1)}^{(4)}(x) &= -x^5, \\ X_{(1)}^{(5)}(x) &= x^4 + \varepsilon x^1x^2. \end{aligned} \tag{2.2}$$

Consequently, via the Theorem 1.4, we assert that the Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^5)$ of the Lorenz atmospheric DEs system (2.1) is described in the following

Corollary 2.2 *(i) The canonical non-linear connection on $J^1(T, \mathbb{R}^5)$ produced by the Lorenz atmospheric DEs system (2.1) has the local components*

$$\hat{\Gamma} = \left(0, \hat{N}_{(1)j}^{(i)} \right),$$

where $\hat{N}_{(1)j}^{(i)}$ are the entries of the matrix

$$\hat{N}_{(1)} = \left(\hat{N}_{(1)j}^{(i)} \right)_{i,j=\overline{1,5}} = \begin{pmatrix} 0 & x^3 - \varepsilon x^5 & 0 & 0 & 0 \\ -x^3 + \varepsilon x^5 & 0 & -x^1 & 0 & \varepsilon x^1 \\ 0 & x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\varepsilon x^1 & 0 & -1 & 0 \end{pmatrix}.$$

(ii) All adapted components of the canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the Lorenz atmospheric DEs system (2.1) vanish.

(iii) All adapted components of the **torsion** d-tensor \hat{T} of the canonical generalized Cartan connection $C\hat{\Gamma}$ **produced by the Lorenz atmospheric DEs system (2.1)** are zero, except

$$\begin{aligned}\hat{R}_{(1)21}^{(3)} &= -\hat{R}_{(1)31}^{(2)} = 1, & \hat{R}_{(1)21}^{(5)} &= -\hat{R}_{(1)51}^{(2)} = -\varepsilon, \\ \hat{R}_{(1)13}^{(2)} &= -\hat{R}_{(1)23}^{(1)} = -1, & \hat{R}_{(1)15}^{(2)} &= -\hat{R}_{(1)25}^{(1)} = \varepsilon.\end{aligned}$$

(iv) All adapted components of the **curvature** d-tensor \hat{R} of the canonical generalized Cartan connection $C\hat{\Gamma}$ **produced by the Lorenz atmospheric DEs system (2.1)** vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the Lorenz atmospheric DEs system (2.1)** has the expression

$$\hat{F} = \hat{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \hat{N}_{(1)k}^{(i)} dx^k, \quad \forall i = \overline{1,5},$$

and the adapted components $\hat{F}_{(i)j}^{(1)}$ are the entries of the matrix

$$\hat{F}^{(1)} = \left(\hat{F}_{(i)j}^{(1)} \right)_{i,j=\overline{1,5}} = -\hat{N}_{(1)}.$$

(vi) The **jet geometric Yang-Mills energy produced by the Lorenz atmospheric DEs system (2.1)** is given by the formula

$$EYM^{Lorenz}(x) = (\varepsilon x^5 - x^3)^2 + (x^1)^2 + (\varepsilon x^1)^2 + 1.$$

Proof. The Lorenz atmospheric DEs system (2.1) is a particular case of the jet first order DEs system (1.1) for $n = 5$ and $X = \left(X_{(1)}^{(i)}(x) \right)_{i=\overline{1,5}}$ given by the relations (2.2). In conclusion, applying the Theorem 1.4, together with the Remark 1.5, and using the Jacobian matrix

$$J(X_{(1)}) = \begin{pmatrix} 0 & -x^3 + \varepsilon x^5 & -x^2 & 0 & \varepsilon x^2 \\ x^3 - \varepsilon x^5 & 0 & x^1 & 0 & -\varepsilon x^1 \\ -x^2 & -x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \varepsilon x^2 & \varepsilon x^1 & 0 & 1 & 0 \end{pmatrix},$$

we obtain what we were looking for. ■

Remark 2.3 Let us remark that, although the jet Yang-Mills electromagnetic energy EYM^{Lorenz} produced by the Lorenz atmospheric DEs system (2.1) depends only by the coordinates x^1 , x^3 and x^5 , it still couples the slow Rossby wave oscillations with the fast gravity wave oscillations. However, the coordinates x^2 and x^4 are missing in the expression of EYM^{Lorenz} . There exists a physical interpretation of this fact?

3 Yang-Mills energetical hypersurfaces of constant level produced by the Lorenz atmospheric DEs system

In the preceding Riemann-Lagrange geometrical theory on the 1-jet space $J^1(T, \mathbb{R}^5)$ the Lorenz atmospheric DEs system (2.1) "produces" a jet Yang-Mills energy given by the formula

$$EYM^{\text{Lorenz}}(x) = (1 + \varepsilon^2) (x^1)^2 + (x^3)^2 + \varepsilon^2 (x^5)^2 - 2\varepsilon x^3 x^5 + 1,$$

where $x = (x^1, x^2, x^3, x^4, x^5)$. In what follows, let us study the *jet Yang-Mills energetical hypersurfaces of constant level* produced by the Lorenz atmospheric DEs system (2.1), which are defined by the implicit equations

$$\Sigma_C^{\text{Lorenz}} : (\varepsilon x^5 - x^3)^2 + (1 + \varepsilon^2) (x^1)^2 = C - 1,$$

where C is a constant real number.

Because Σ_C^{Lorenz} is a *quadric* in the system of axes $Ox^1x^3x^5$ for every $C \in \mathbb{R}$, then, using the reduction to the canonical form of a quadric, we find the following geometrical results:

1. If $C < 1$, then we have $\Sigma_{C < 1}^{\text{Lorenz}} = \emptyset$;
2. If $C = 1$, then we have

$$\Sigma_{C=1}^{\text{Lorenz}} : \begin{cases} x^1 = 0 \\ x^3 - \varepsilon x^5 = 0, \end{cases}$$

that is $\Sigma_{C=1}^{\text{Lorenz}}$ is a *straight line* in the system of axes $Ox^1x^3x^5$;

3. If $C > 1$, then we have

$$\Sigma_{C>1}^{\text{Lorenz}} : (1 + \varepsilon^2) (x^1)^2 + (x^3)^2 + \varepsilon^2 (x^5)^2 - 2\varepsilon x^3 x^5 - C + 1 = 0,$$

that is $\Sigma_{C>1}^{\text{Lorenz}}$ is a degenerate non-empty quadric in the system of axes $Ox^1x^3x^5$, whose canonical form is

$$\Sigma_{C>1}^{\text{Lorenz}} : (X^3)^2 + (X^5)^2 = \frac{C-1}{1+\varepsilon^2},$$

where the rotation of the system of axes $Ox^1x^3x^5$ into the system of axes $OX^1X^3X^5$ is given by the matriceal relation

$$\begin{pmatrix} x^1 \\ x^3 \\ x^5 \end{pmatrix} = \frac{1}{\sqrt{1+\varepsilon^2}} \begin{pmatrix} 0 & \sqrt{1+\varepsilon^2} & 0 \\ \varepsilon & 0 & 1 \\ 1 & 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} X^1 \\ X^3 \\ X^5 \end{pmatrix}.$$

In conclusion, the degenerate non-empty quadric $\Sigma_{C>1}^{\text{Lorenz}}$ is in the system of axes $Ox^1x^3x^5$ a *slant circular cylinder* of radius

$$R = \sqrt{\frac{C-1}{1+\varepsilon^2}},$$

having as axis of symmetry the straight line $\Sigma_{C=1}^{\text{Lorenz}}$.

Open problem. There exist real physical interpretations, in the study of the Lorenz atmospheric DEs system (2.1), for the preceding geometrical results?

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